Linear Algebra - Midterm Exam

December 14, 2020 17.30-19.30 Online exam

Read carefully the exam rules on the Nestor page of the exam
Remember that you are not allowed to use the internet during the exam. The only exception is if you use an online text editor (e.g., overleaf) to write the exam. But notice that you are not allowed to use an online editor if it has tools to solve math problems.

QUESTIONS:

1. (a) $\boxed{1.5}$ You are given two 3×2 matrices, B and C, with, respectively, elements b_{ik} and c_{ik}, and two column vectors, $\mathbf{x}$ and $\mathbf{y}$ with, respectively, components x_i and y_i, $i = 1, 2, 3$, $k = 1, 2$.

   Find the elements, a_{ij}, of a 3×3 matrix, A, such that $AB = C$, and $A\mathbf{x} = \mathbf{y}$. Remember, B, C, $\mathbf{x}$ and $\mathbf{y}$ are known, and A is unknown. Finally, if necessary, assume that all square matrices in this exercise are invertible.

   Hint: First write each of the components of the matrix/matrix product $AB = C$ and the matrix/vector product $A\mathbf{x} = \mathbf{y}$ as a system of linear equations. By looking at those equations construct two new matrices, one called B', using the elements of B and $\mathbf{x}$, and the other one called C', using the elements of C and $\mathbf{y}$, such that a single matrix product, $AB' = C'$, gives exactly the same system of linear equations. Once you have the matrix product, explain with words (no need to do any complicated calculation) how you would get A.

   (b) $\boxed{0.5}$ What are the elements of A when $B = \begin{bmatrix} 1 & 0 \\ -5 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 1 \\ 5 & 5 \\ 1 & -1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$?

2. Check whether in each case the given vectors form a basis in the corresponding vector space. Justify your answer.

   (a) $\boxed{1}$ $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 5 \\ -9 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ -7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 5 \\ 8 \end{bmatrix}$.

   If the answer is yes, give the transition matrices from, respectively, the basis $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to the basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and from the basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the basis $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and write the coordinate vector $[\mathbf{x}]_E$ given in the basis E as a coordinate vector $[\mathbf{x}]_V$ in the basis V:

   $$[\mathbf{x}]_E = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_E$$

   (b) $\boxed{1}$ $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ -6 \\ 5 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

   If the answer is yes, give the transition matrices from, respectively, the basis $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ to the basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, and from the basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ to the basis $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, and write the coordinate vector $[\mathbf{x}]_E$ given in the basis E as a coordinate vector $[\mathbf{x}]_V$ in the basis V:

   $$[\mathbf{x}]_E = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 0 \end{bmatrix}_E$$

Please turn over

3. (a) $\boxed{0.5}$ Show that the following two vectors are linearly independent:

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} -5/2 \\ 0 \\ 1 \end{bmatrix}.$$

(b) $\boxed{0.5}$ Find the general form of any vector that is a linear combination of these two vectors. This general new vector gives the equation of a plane. (The head of each particular vector represented by this general vector is a point in that plane.).

(c) $\boxed{1}$ Write a general vector in the direction perpendicular to this plane. Write that vector in parametric form. Write also the unit vector in the direction perpendicular to this plane.
Hint: You may want to select 2 out of the infinite vectors that define the plane.

4. (a) $\boxed{1}$ Show that the vectors $f_1(x) = x$, $f_2(x) = \sin x$ and $f_3(x) = \cos x$ in the vector space $C^2[0, \pi]$ (the set of all the functions that have a continuous second derivatives on the interval $[0, \pi]$) are linearly independent.

(b) $\boxed{1}$ Given the vectors $p(x) = a$, $q(x) = 2x + 4$ and $r(x) = (a - 1)x^2$ in the vector space P_3 of all polynomials with degree < 3, find for which values of a the three vectors are linearly independent.

5. Consider 3 libraries, one in Groningen, one in Assen and one in Haren. Books are borrowed in the morning, and returned in the evening. A book may be returned at each of the 3 libraries. The books are borrowed and returned in the following way:

From the books borrowed in Groningen 60% are returned in Groningen, 20% are returned in Assen and 20% are returned in Haren.
From the books borrowed in Assen 70% are returned in Assen, none is returned in Groningen and 30% are returned in Haren.
All the books borrowed in Haren are returned in Haren.
The fraction of books borrowed and returned at each library each day is the same, and the total number of books is constant.

On Tuesday Dec. 8 2020 there were 600 books in Groningen, 200 in Assen and 500 in Haren.

(a) $\boxed{1}$ Determine the numbers of books at each library two days later, on Dec. 10 2020.

(b) $\boxed{1}$ Determine the numbers of books at each library one day earlier, on Dec. 7 2020.